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One-loop n -point equivalence among negative-dimensional, Mellin–Barnes and Feynman parametrization approaches to Feynman integrals

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Abstract

We show that at one-loop order, negative-dimensional, Mellin–Barnes' (MB) and Feynman parametrization (FP) approaches to Feynman loop integral calculations are equivalent. Starting with a generating functional, for two and then for n -point scalar integrals, we show how to reobtain MB results, using negative-dimensional and FP techniques. The n -point result is valid for different masses, arbitrary exponents of propagators and dimension.

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1. Introduction

The amazing comparison [1] between experimental determination and theoretical prediction of the anomalous magnetic momentum of the electron is the greatest motivation, in our opinion, to study and develop techniques that allow the precise calculation of higher order Feynman loop integrals. Recently there has been increasing interest in studying processes such as [2] $e^+e^- \rightarrow 3$ jets and $e^+e^- \rightarrow 4$ jets, so loop integrals with five and six external legs must be known.

The physicists' battle against the tricky Feynman loop integrals is fought on many fronts. We can cite some of them: integration by parts method [3] seems to be the most powerful one, since one can in most cases reduce the number of loops, e.g., the scalar massless two-loop master can be rewritten as a sum of two simpler integrals: a two-loop self-energy with an insertion [4, 5] plus the square of a one-loop self-energy. This is a very simple example—where one must deal with a greater number of simpler integrals with powers of propagators shifted—of a powerful technique, see for instance a five-loop calculation in [6].

Integration by parts is also associated with other methods. In fact, one cannot evaluate a Feynman loop integral using the above-mentioned technique alone. It simplifies the original diagram but does not solve it. In order to carry out the integration, Gehrmann and Remiddi [7] used the differential equation method, introduced by Kotikov [8]³, and solved a large class of difficult problems. Glover and collaborators completed the study of the whole class $2 \rightarrow 2$ of two-loop scattering [9]. Also, the Gegenbauer polynomial method has been used in order to study a complicated process [10].

Other methods that make use of the decomposition of complicated integrals, such as the one-loop pentagon [11], were developed as well as string inspired ones [12]. See also [13–16] for other important approaches, a very powerful numerical technique on [17] and for a review of the progress of loop calculations [18].

The Mellin–Barnes (MB) and the negative-dimensional integration method (NDIM) are two other interesting and powerful techniques to tackle such Feynman integrals. The Mellin–Barnes approach relies on the relation

$$\left(\sum_{i=1}^n z_i\right)^{-B} = \frac{1}{z_1^B (2\pi i)^{n-1} \Gamma(B)} \int_{-i\infty}^{i\infty} \Gamma(w_1 + w_2 + \cdots + w_{n-1} + B) \times \prod_{i=1}^{n-1} \left[dw_i \left(\frac{z_{i+1}}{z_1}\right)^{w_i} \Gamma(-w_i) \right]. \quad (1)$$

In other words, we rewrite each propagator as a Mellin transform. However, these parametric integrals are not difficult to solve—as happens in the Feynman parametrization approach where the integrals become more and more complicated—because one can apply the Cauchy theorem and two Barnes lemmas [19]. The MB approach is greatly used by Tausk [20], Smirnov [21], Davydychev [22, 23] and co-workers in order to tackle two- and three-loop integrals. The results are always expressed as generalized hypergeometric functions which depend on adimensional ratios of momenta and/or masses, spacetime dimension D and exponents of propagators.

On the other hand, NDIM is a technique whereby it is not necessary to introduce parametric integrals, the Feynman integral is the result of the comparison between two calculations: a Gaussian-like integral (the generating functional of negative-dimensional integrals) and a Taylor expansion of the generating functional [24]. It is worth mentioning that in the NDIM context performing the calculation for a particular set of exponents of propagators presents the same difficulties as performing the same task for arbitrary values of them. The results are, just as in the MB approach, given in terms of generalized hypergeometric functions which depend on the same quantities mentioned above.

One can then rightfully ask: is there any connection between these two approaches? The answer is yes. The purpose of our paper is to show that they are equivalent, at least at one-loop order. In fact, one can argue that the results must be the same if one correctly applied both methods, and we will show this explicitly. However, it can be useful and interesting to study which of them is more powerful when the number of external legs increases. On the one hand NDIM demands computer facilities in order to solve a large number of systems and browse the large number of results; on the other hand, MB does not require computers but the integrals must be calculated one by one. Another point to observe is that NDIM relies on Grassmannian integrals and MB on Mellin transforms, i.e., apparently different (as far as we know) subjects. Also, showing the equivalence between them and knowing the routes which take us to NDIM or to MB one can build, for instance, a technique like NDIM in order to tackle problems in

³ The last paper also deals with n -point integrals.

finite temperature field theory, such as calculations of heat-kernel which can be dealt with using Mellin integrals.

The outline of our paper is as follows: in section 2 we present a step-by-step calculation of the two-point scalar integral starting with the NDIM approach and arriving at an expression originally obtained by Davydychev [23] using the MB approach, then we repeat the same process using FP. In the next section, we deal with an arbitrary number N of external legs, also starting in the negative-dimensional approach and showing how to obtain Davydychev's original result calculated in the MB scheme; we carry out the same for the integrals with FP. In the final section, we present our conclusions and a discussion concerning the three methods.

2. One-loop two-point function

In this section we present the calculations to evaluate the one-loop two-point scalar integral within the NDIM scheme and compare this result with that obtained by the MB approach. Consider the integral,

$$I = \int d^D k \exp \{ -\alpha [k^2 - m_1^2] - \beta [(k - q)^2 - m_2^2] \} \quad (2)$$

which is the usual generating functional for negative-dimensional integrals. We will always begin with this kind of generating functional, for two- and n -point scalar integrals, and after some manipulations arrive at results, which were obtained previously by other authors, using the MB approach.

The first step in the NDIM context is a series expansion of the above integral,

$$I = \sum_{a_1, a_2=0}^{\infty} \frac{(-\alpha)^{a_1} (-\beta)^{a_2}}{a_1! a_2!} J^{(2)}(a_1, a_2; q; m_1, m_2) \quad (3)$$

where we define the negative-dimensional integrals,

$$\begin{aligned} J^{(2)} &= J^{(2)}(a_1, a_2; q; m_1, m_2) \\ &= \int d^D k [k^2 - m_1^2]^{a_1} [(k - q)^2 - m_2^2]^{a_2}. \end{aligned} \quad (4)$$

The integration (2) can be easily done,

$$I = \left(\frac{\pi}{\alpha + \beta} \right)^{D/2} \exp \left\{ - \left(\frac{\alpha\beta}{\alpha + \beta} \right) q^2 + \alpha m_1^2 + \beta m_2^2 \right\} \quad (5)$$

and the exponential above expanded again in Taylor series,

$$I = \left(\frac{\pi}{\alpha + \beta} \right)^{D/2} \sum_{j_0=0}^{\infty} \frac{1}{j_0!} \left[- \left(\frac{\alpha\beta}{\alpha + \beta} \right) q^2 + \alpha m_1^2 + \beta m_2^2 \right]^{j_0}.$$

Rewriting it as

$$I = \pi^{D/2} \sum_{j_0=0}^{\infty} \frac{(\alpha + \beta)^{-D/2+j_0} (m_2^2)^{j_0}}{j_0!} \left[1 - \frac{\alpha\beta}{(\alpha + \beta)^2} \frac{q^2}{m_2^2} - \frac{\alpha}{\alpha + \beta} \left(1 - \frac{m_1^2}{m_2^2} \right) \right]^{j_0} \quad (6)$$

a multinomial expansion gives us

$$I = \pi^{D/2} \sum_{j_0, b_1, c_1=0}^{\infty} \frac{\alpha^{b_1+c_1} \beta^{b_1} (\alpha + \beta)^{-D/2+j_0-2b_1-c_1} (m_2^2)^{j_0} \Gamma(-j_0 + b_1 + c_1) \left(\frac{q^2}{m_2^2} \right)^{b_1} \left(1 - \frac{m_1^2}{m_2^2} \right)^{c_1}}{\Gamma(1 + j_0) \Gamma(-j_0) b_1! c_1!} \quad (7)$$

and using

$$\left[\sum_{i=1}^n \alpha_i \right]^{-A} = \frac{1}{\alpha_1^A} \sum_{j_1, \dots, j_{n-1}=0}^{\infty} \frac{(A)_{j_1} (-j_1)_{j_2} (-j_2)_{j_3} \dots (-j_{n-2})_{j_{n-1}}}{(1)_{j_1} (1)_{j_2} (1)_{j_3} \dots (1)_{j_{n-1}}} \times \left(-\frac{\alpha_2}{\alpha_1} \right)^{j_1} \left(-\frac{\alpha_3}{\alpha_2} \right)^{j_2} \left(-\frac{\alpha_4}{\alpha_3} \right)^{j_3} \dots \left(-\frac{\alpha_n}{\alpha_{n-1}} \right)^{j_{n-1}} \quad (8)$$

in the factor $(\alpha + \beta)^{-A}$, one obtains

$$I = \pi^{D/2} \sum_{j_0, j_1=0}^{\infty} \sum_{b_1, c_1=0}^{\infty} \alpha^{-D/2+j_0-b_1-j_1} \beta^{b_1+j_1} \times \frac{(-1)^{j_1} (m_2^2)^{j_0} \Gamma(-j_0+b_1+c_1) \Gamma(D/2-j_0+2b_1+c_1+j_1)}{\Gamma(1+j_0) \Gamma(-j_0) \Gamma(D/2-j_0+2b_1+c_1) \gamma(1+j_1)} \times \left(\frac{q^2}{m_2^2} \right)^{b_1} \left(1 - \frac{m_1^2}{m_2^2} \right)^{c_1} \frac{1}{b_1! c_1!} \quad (9)$$

where $(a)_b = \Gamma(a+b)/\Gamma(a)$ is the Pochhammer symbol.

Comparing the power of the parameters α and β between (3) and (9), we have the following constraint equations:

$$a_1 = -D/2 + j_0 - b_1 - j_1 \quad (10)$$

$$a_2 = b_1 + j_1 \quad (11)$$

and, after solving for j_0 and j_1 , we have

$$j_0 = a_1 + a_2 + D/2 = \sigma_2 \quad (12)$$

$$j_1 = a_2 - b_1. \quad (13)$$

Performing the substitution of this result in (9) and the analytic continuation to $a_1, a_2 \leq 0$, we arrive at

$$J^{(2)} = \pi^{D/2} (-m_2^2)^{\sigma_2} \frac{\Gamma(-\sigma_2)}{\Gamma(-a_1 - a_2)} \sum_{b_1, c_1=0}^{\infty} \frac{(-\sigma_2)_{b_1+c_1} (-a_1)_{b_1+c_1} (-a_2)_{b_1}}{b_1! c_1! (-a_1 - a_2)_{2b_1+c_1}} \left(\frac{q^2}{m_2^2} \right)^{b_1} \left(1 - \frac{m_1^2}{m_2^2} \right)^{c_1} \quad (14)$$

which is the well-known result for two-point scalar integrals with different masses, see [19].

2.1. Two-point function via Feynman parametrization

The most popular technique to deal with loop integrals is certainly Feynman parametrization. It is the one the students learn on field theory courses, and one of the few that textbooks introduce (another is α -parametrization). Depending on the manipulations one performs, it can turn the original loop integrals into a hefty one. We will proceed in a slightly different route. Our aim is to show how one can obtain the previous results for two-point functions, given in terms of hypergeometric functions, using FP since in most cases the results calculated through FP are written as polylogarithms $Li_n(z)$, $n = 0, 1, 2, 3, 4$.

Hypergeometric functions have an advantage over dilogarithms, for instance, in the case of photon–photon scattering scalar integrals. The result for $|s/4m^2| < 1$, $|t/4m^2| < 1$ can be written as a single Appel function F_3 of two variables and five parameters: on the other hand,

the same result can be recast as a sum of several $Li_2(z_j)$ functions of complicated arguments z_j , see for instance [25].

Consider the function

$$\begin{aligned}
 F^{(2)} &= F^{(2)}(a_1, a_2; q; m_1; m_2; x_0, x_1) \\
 &= \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^{x_0} dx_1 (x_0 - x_1)^{a_1-1} (x_1 - x_2)^{a_2-1} \\
 &\quad \times \int \frac{d^D k}{\{[k^2 - m_1^2](x_0 - x_1) + [(k - q)^2 - m_2^2](x_1 - x_2)\}^{a_1+a_2}}
 \end{aligned}$$

where $a_1, a_2 \geq 0$. We note that when $x_0 = 1, x_2 = 0$ we have the well-known Feynman parametrization to the propagator of $J^{(2)}$, that is $F^{(2)} = J^{(2)}$. Such modification will turn the calculation of n -point integrals simpler as shown in section 3.1. This expression can be rewritten as follows:

$$\begin{aligned}
 F^{(2)} &= \frac{\Gamma(a_1 + a_2)(x_0 - x_2)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^{x_0} dx_1 (x_0 - x_1)^{a_1-1} (x_1 - x_2)^{a_2-1} \\
 &\quad \times \int d^D k \left\{ \left[k - \frac{q(x_1 - x_2)}{(x_0 - x_2)^{1/2}} \right]^2 - m_2^2(x_0 - x_2) \right. \\
 &\quad \left. + \frac{q^2(x_0 - x_1)(x_1 - x_2)}{x_0 - x_2} + (m_2^2 - m_1^2)(x_0 - x_1) \right\}^{-a_1-a_2}
 \end{aligned}$$

and after the evaluation of the integral in k using the well-known formula,

$$\int d^D k \frac{(k^2)^\alpha}{(k^2 + M^2)^\beta} = \pi^{D/2} (M^2)^{\alpha+D/2-\beta} \frac{\Gamma(\beta - \alpha - D/2)\Gamma(\alpha + D/2)}{\Gamma(\beta)\Gamma(D/2)} \tag{15}$$

we have

$$\begin{aligned}
 F^{(2)} &= \pi^{D/2} \frac{\Gamma(a_1 + a_2 - D/2)(x_0 - x_2)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^{x_0} dx_1 (x_0 - x_1)^{a_1-1} (x_1 - x_2)^{a_2-1} \\
 &\quad \times \left\{ -m_2^2(x_0 - x_2) + \frac{q^2(x_0 - x_1)(x_1 - x_2)}{x_0 - x_2} + (m_2^2 - m_1^2)(x_0 - x_1) \right\}^{D/2-a_1-a_2}.
 \end{aligned}$$

Taylor expanding, we get

$$\begin{aligned}
 F^{(2)} &= \pi^{D/2} (-m_2^2)^{D/2-a_1-a_2} \frac{(x_0 - x_2)^{-a_1-a_2}}{\Gamma(a_1)\Gamma(a_2)} \\
 &\quad \times \sum_{b_1=0}^{\infty} \sum_{c_1=0}^{\infty} (x_0 - x_2)^{-b_1} x_0^{-b_1-c_1} \Gamma(a_1 + a_2 - D/2 + b_1 + c_1) \\
 &\quad \times \frac{1}{b_1!} \left[\frac{q^2}{m_2^2} \right]^{b_1} \frac{1}{c_1!} \left[1 - \frac{m_1^2}{m_2^2} \right]^{c_1} \int_0^{x_0} dx_1 (x_0 - x_1)^{a_1+b_1+c_1-1} (x_1 - x_2)^{a_2+b_1-1}.
 \end{aligned}$$

These integrals in x_1 can be evaluated, with $x_0 = 1, x_2 = 0$, using

$$\prod_{i=0}^{n-2} \left[\int_0^{x_i} dx_{i+1} (x_i - x_{i+1})^{\alpha_{i+1}-1} \right] x_{n-1}^{\alpha_n-1} = x_0^{\alpha_1+\dots+\alpha_n-1} \frac{\prod_{i=1}^n \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^n \alpha_i)}. \tag{16}$$

One obtains

$$\begin{aligned}
 F^{(2)} &= \pi^{D/2} (-m_2^2)^{D/2-a_1-a_2} \frac{\Gamma(a_1 + a_2 - D/2)}{\Gamma(a_1 + a_2)} \sum_{b_1=0}^{\infty} \sum_{c_1=0}^{\infty} \frac{(a_1 + a_2 - D/2)_{b_1+c_1} (a_1)_{b_1+c_1} (a_2)_{b_1}}{b_1! c_1! (a_1 + a_2)_{2b_1+c_1}} \\
 &\quad \times \left(\frac{q^2}{m_2^2} \right)^{b_1} \left(1 - \frac{m_1^2}{m_2^2} \right)^{c_1} \tag{17}
 \end{aligned}$$

which is exactly the former result (14). This result shows that, to one-loop two-point, the NDIM, Feynman parametrization and Mellin–Barnes (MB) representation are equivalent. The other kinematical regions can be obtained through analytic continuation of the hypergeometric function above (see [19, 22, 26]).

3. n -point function

In this section we present the generalization of the previous ideas in order to obtain the MB result for the scalar integral associated with n -point function. We consider a one-loop Feynman diagram with n external legs with momenta: $p_1 = l_2 - l_1, p_2 = l_3 - l_2, \dots, p_n = l_1 - l_n$, and internal momenta $k - l_1, k - l_2, \dots, k - l_n$. From a similar reasoning, we begin with the generating functional

$$I_n = \int d^D k \exp \left\{ - \sum_{i=1}^n \alpha_i [(k - l_i)^2 - m_i^2] \right\} \quad (18)$$

$$\begin{aligned} &= \int d^D k \prod_{i=1}^n \sum_{a_i=0}^{\infty} \frac{(-\alpha_i)^{a_i}}{a_i!} [(k - l_i)^2 - m_i^2]^{a_i} \\ &= \sum_{a_1, \dots, a_n=0}^{\infty} \frac{(-\alpha_1)^{a_1}}{a_1!} \dots \frac{(-\alpha_n)^{a_n}}{a_n!} J^{(n)}(l_1, m_1, a_1; \dots; l_n, m_n, a_n) \end{aligned} \quad (19)$$

where $J^{(n)}(l_1, m_1, a_1; \dots; l_n, m_n, a_n)$ represents the n -point functions for negative values of a_i and is given by

$$\begin{aligned} J^{(n)} &= J^{(n)}(l_1, m_1, a_1; \dots; l_n, m_n, a_n) \\ &= \int d^D k \prod_{i=1}^n [(k - l_i)^2 - m_i^2]^{a_i}. \end{aligned} \quad (20)$$

Expression (19), after the integration in k , can be rewritten in the form

$$I_n = \left(\frac{\pi}{\sum_{i=1}^n \alpha_i} \right)^{D/2} \exp \left\{ - \frac{\sum_{i>j}^n \alpha_i \alpha_j l_{ij}^2}{\sum_{i=1}^n \alpha_i} + \sum_{i=1}^n \alpha_i m_i^2 \right\}$$

where $l_{ij} = l_i - l_j$. After a new expansion on the right-hand side of the above expression, we have

$$I_n = \left(\frac{\pi}{\sum_{i=1}^n \alpha_i} \right)^{D/2} \sum_{j_0=0}^{\infty} \frac{1}{\Gamma(1 + j_0)} \left[- \frac{\sum_{i>j}^n \alpha_i \alpha_j l_{ij}^2}{\sum_{i=1}^n \alpha_i} + \sum_{i=1}^n \alpha_i m_i^2 \right]^{j_0}. \quad (21)$$

Using expansions (1) and (8) for $[\sum_{i=1}^n \alpha_i]$ multinomial, with $N = n(n-1)/2$ terms, we get

$$\begin{aligned} I_n &= \pi^{D/2} \frac{1}{(2\pi i)^{N+n-1}} \sum_{j_0, \dots, j_{n-1}=0}^{\infty} \frac{(-l_{12}^2)^{j_0}}{\Gamma(1 + j_0) \Gamma(-j_0)} \int_{-i\infty}^{i\infty} \frac{(-1)^{j_1}}{\Gamma(1 + j_1)} \prod_{i=2}^{n-1} \left[\frac{(-1)^{j_i} \Gamma(j_i - j_{i-1})}{\Gamma(1 + j_i) \Gamma(-j_{i-1})} \right] \\ &\times \frac{\Gamma(\sum_{j>1, i<j} w_{ij} + \sum_{i=1}^n v_i - j_0) \Gamma(D/2 + j_0 + j_1 - \sum_{i=1}^n v_i)}{\Gamma(D/2 + j_0 - \sum_{i=1}^n v_i)} \\ &\times \prod_{j>2, i<j}^n \left[dw_{ij} \left(\frac{l_{ij}^2}{l_{12}^2} \right)^{w_{ij}} \Gamma(-w_{ij}) \right] \prod_{i=1}^n \left[dv_i \left(-\frac{m_i^2}{l_{12}^2} \right)^{v_i} \Gamma(-v_i) \right] \prod_i^n [\alpha_i^{f_i}] \end{aligned} \quad (22)$$

where

$$\begin{aligned}
 f_1 &= -j_1 - D/2 - \sum_{i \neq 1, i < j}^n w_{ij} + v_1 \\
 f_2 &= j_0 + j_1 - j_2 - \sum_{i \neq 2, i < j}^n w_{ij} - \sum_{i \neq 2}^n v_i \\
 f_i &= j_{i-1} - j_i + \sum_{j \neq i}^n w_{ij} + v_i \quad i = 3, 4, \dots, n - 1 \\
 f_n &= j_{n-1} + \sum_{j \neq n}^n w_{nj} + v_n.
 \end{aligned}
 \tag{23}$$

We now need to compare, term by term, α_i powers in equation (22) with those of equation (19). We obtain $f_i = a_i$ and the solution of the above system will be given by

$$\begin{aligned}
 j_0 &= \sum_{i=1}^n a_i = \sigma_n \\
 j_1 &= -a_1 - D/2 - \sum_{i \neq 1, i < j}^n w_{ij} + v_1 \\
 j_2 - j_1 &= -a_2 + \sigma_n - \sum_{i \neq 2, i < j}^n w_{ij} - \sum_{i \neq 2}^n v_i \\
 j_i - j_{i-1} &= -a_i + \sum_{j \neq i}^n w_{ij} + v_i \quad i = 3, 4, \dots, n - 1 \\
 j_{n-1} &= a_n - \sum_{j \neq n}^n w_{nj} - v_n.
 \end{aligned}
 \tag{24}$$

Substituting the above solutions into (22), we arrive at

$$\begin{aligned}
 J^{(n)} &= \pi^{D/2} (l_{12}^2)^{\sigma_n} \frac{(-1)^{a_1+\dots+a_n} (1)_{a_1+\epsilon} \cdots (1)_{a_n+\epsilon}}{(2\pi i)^{N+n-1} [\Gamma(1+\epsilon)\Gamma(-\epsilon)]^n} \int_{-i\infty}^{i\infty} \prod_{j>2, i<j}^n \left[dw_{ij} \left(\frac{l_{ij}^2}{l_{12}^2} \right)^{w_{ij}} \Gamma(-w_{ij}) \right] \\
 &\times \prod_{i=1}^n \left[dv_i \left(-\frac{m_i^2}{l_{12}^2} \right)^{v_i} \Gamma(-v_i) \right] \prod_{i=3}^n \left[\Gamma \left(-a_i + \sum_{j \neq i}^n w_{ij} - \sum_{i \neq 2}^n v_i \right) \right] \\
 &\times \frac{\Gamma(\sum_{j>1, i<j}^n w_{ij} + \sum_{i=1}^n v_i - \sigma_n) \Gamma(\sigma_n - a_1 - \sum_{i \neq 1, i < j}^n w_{ij} - \sum_{i=2}^n v_i)}{\Gamma(D/2 + \sigma_n - \sum_{i=1}^n v_i)} \\
 &\times \Gamma \left(-a_2 + \sigma_n - \sum_{i \neq 2, i < j}^n w_{ij} - \sum_{i \neq 2}^n v_i \right)
 \end{aligned}
 \tag{25}$$

that after carrying out analytic continuation to negative values of the a_i provides

$$\begin{aligned}
 J^{(n)} &= \pi^{D/2} (l_{12}^2)^{\sigma_n} \frac{1}{(2\pi i)^{N+n-1}} \prod_{i=1}^n \left[\frac{1}{\Gamma(-a_i)} \right] \\
 &\times \int_{-i\infty}^{i\infty} \prod_{j>2, i<j}^n \left[dw_{ij} \left(\frac{l_{ij}^2}{l_{12}^2} \right)^{w_{ij}} \Gamma(-w_{ij}) \right] \prod_{i=1}^n \left[dv_i \left(-\frac{m_i^2}{l_{12}^2} \right)^{v_i} \Gamma(-v_i) \right]
 \end{aligned}$$

$$\begin{aligned} & \times \prod_{i=3}^n \left[\Gamma \left(-a_i + \sum_{j \neq i}^n w_{ij} - \sum_{i \neq 2}^n v_i \right) \right] \Gamma \left(-a_2 + \sigma_n - \sum_{i \neq 2, i < j}^n w_{ij} - \sum_{i \neq 2}^n v_i \right) \\ & \times \frac{\Gamma \left(\sum_{j > 1, i < j}^n w_{ij} + \sum_{i=1}^n v_i - \sigma_n \right) \Gamma \left(\sigma_n - a_1 - \sum_{i \neq 1, i < j}^n w_{ij} - \sum_{i=2}^n v_i \right)}{\Gamma \left(D/2 + \sigma_n - \sum_{i=1}^n v_i \right)}. \end{aligned} \tag{26}$$

The above result is also an expression for the n -point scalar integrals with arbitrary exponents of propagators and dimension, in the MB scheme. However, it was not the one obtained by Davydychev in [23]. Formula (26) is a new result.

It is important to observe that the above result, equation (26), is valid since the series is convergent which means $|l_{ij}^2/l_{12}^2| < 1$ and $|m_i^2/l_{12}^2| < 1$, that is, external momentum is greater than masses. Conversely, Davydychev’s result holds in another kinematical region, namely, where $|l_{ij}^2/m_n^2| < 1$ and $|1 - m_i^2/m_n^2| < 1$, i.e. when masses are greater than incoming/outcoming momenta. This result could be obtained, in principle, from Davydychev’s formula through analytic continuation. However, we stress the point that such analytic continuation formulae are not known for multiple hypergeometric series (in general these formulae are known only in the case of single and double series).

Another form to represent the n -point function can be obtained also from expansion of (21), that is

$$\begin{aligned} I_n &= \pi^{D/2} \frac{1}{(2\pi i)^{N+n-1}} \sum_{j_0, \dots, j_{n-1}=0}^{\infty} \frac{(m_n^2)^{j_0}}{\Gamma(1+j_0)\Gamma(-j_0)} \int_{-i\infty}^{i\infty} \frac{(-1)^{j_1}}{\Gamma(1+j_1)} \prod_{i=2}^{n-1} \left[\frac{(-1)^{j_i} \Gamma(j_i - j_{i-1})}{\Gamma(1+j_i)\Gamma(-j_{i-1})} \right] \\ & \times \frac{\Gamma \left(\sum_{i < j}^n w_{ij} + \sum_{i=1}^{n-1} v_i - j_0 \right) \Gamma \left(D/2 + j_1 + \sum_{i, j \neq n, i < j}^n w_{ij} \right)}{\Gamma \left(D/2 + \sum_{i, j \neq n, i < j}^n w_{ij} \right)} \\ & \times \prod_{i < j}^n \left[dw_{ij} \left(-\frac{l_{ij}^2}{m_n^2} \right)^{w_{ij}} \Gamma(-w_{ij}) \right] \prod_{i=1}^{n-1} \left[dv_i \left(\frac{m_i^2}{m_n^2} \right)^{v_i} \Gamma(-v_i) \right] \prod_i [\alpha_i^{g_i}] \end{aligned} \tag{27}$$

where

$$\begin{aligned} g_i &= j_i - j_{i+1} + \sum_{j \neq i}^n w_{ij} + v_i \quad i = 1, 2, \dots, n-2 \\ g_{n-1} &= j_{n-1} + \sum_{j \neq n-1}^n w_{n-1, j} + v_{n-1} \\ g_n &= j_0 - j_1 - D/2 - 2 \sum_{i, j \neq n, i < j}^n w_{ij} - \sum_{i=1}^{n-1} v_i \end{aligned} \tag{28}$$

whose solution, after comparing the α_i powers of equation (27) with (19), $g_i = a_i$, is given by

$$\begin{aligned} j_0 &= \sum_{i=1}^n a_i = \sigma_n \\ j_1 &= \sigma_n - D/2 - 2 \sum_{i, j \neq n, i < j}^n w_{ij} - \sum_{i=1}^{n-1} v_i \end{aligned}$$

$$\begin{aligned}
 j_{i+1} - j_i &= -a_i + \sum_{j \neq i}^n w_{ij} + v_i & i = 1, 2, \dots, n-2 \\
 j_{n-1} &= a_{n-1} - \sum_{j \neq n-1}^n w_{n-1,j} - v_{n-1}.
 \end{aligned}
 \tag{29}$$

Analytically continuing and substituting the above result in (27) and comparing its α_i powers with (19), we arrive at

$$\begin{aligned}
 J^{(n)} &= \pi^{D/2} (-m_n^2)^{\sigma_n} \frac{(-1)^{a_1+\dots+a_n} (1)_{a_1+\epsilon} \dots (1)_{a_n+\epsilon}}{(2\pi i)^{N+n-1} [\Gamma(1+\epsilon)\Gamma(-\epsilon)]^n} \int_{-i\infty}^{i\infty} \prod_{i=1}^{n-1} \left[\Gamma \left(-a_i + \sum_{j \neq i}^n w_{ij} + v_i \right) \right] \\
 &\times \frac{\Gamma(\sum_{i < j}^n w_{ij} + \sum_{i=1}^{n-1} v_i - \sigma_n) \Gamma(\sigma_n - \sum_{i,j \neq n, i < j}^n w_{ij} - \sum_{i=1}^{n-1} v_i)}{\Gamma(D/2 + \sum_{i,j \neq n, i < j}^n w_{ij})} \\
 &\times \prod_{i < j}^n \left[dw_{ij} \left(-\frac{l_{ij}^2}{m_n^2} \right)^{w_{ij}} \Gamma(-w_{ij}) \right] \prod_{i=1}^{n-1} \left[dv_i \left(\frac{m_i^2}{m_n^2} \right)^{v_i} \Gamma(-v_i) \right]
 \end{aligned}
 \tag{30}$$

that can be rewritten in the form

$$\begin{aligned}
 J^{(n)} &= \pi^{D/2} (-m_n^2)^{\sigma_n} \frac{(-1)^{a_1+\dots+a-n} (1)_{a_1+\epsilon} \dots (1)_{a_n+\epsilon}}{(2\pi i)^{N+n-1} [\Gamma(1+\epsilon)\Gamma(-\epsilon)]^n} \int_{-i\infty}^{i\infty} \prod_{i < j}^n \left[dw_{ij} \left(-\frac{l_{ij}^2}{m_n^2} \right)^{w_{ij}} \Gamma(-w_{ij}) \right] \\
 &\times \prod_{i=1}^{n-1} \left[dv_i \left(\frac{m_i^2}{m_n^2} \right)^{v_i} \Gamma(-v_i) \Gamma \left(-a_i + \sum_{j \neq i}^n w_{ij} + v_i \right) \right] \\
 &\times \frac{\Gamma(\sum_{i < j}^n w_{ij} + \sum_{i=1}^{n-1} v_i - \sigma_n) \Gamma(\sigma_n - \sum_{i,j \neq n, i < j}^n w_{ij} - \sum_{i=1}^{n-1} v_i)}{\Gamma(D/2 + \sum_{i,j \neq n, i < j}^n w_{ij})},
 \end{aligned}
 \tag{31}$$

that after the analytic continuation to negative values of a_i gives

$$\begin{aligned}
 J^{(n)} &= \pi^{D/2} (-m_n^2)^{\sigma_n} \frac{\Gamma(-\sigma_n)}{\Gamma(-\sigma_n + D/2)} \sum_{b_{ij}=0}^{\infty} \sum_{c_l=0}^{\infty} \\
 &\times \frac{(-\sigma_n)_{\sum_{i>j} b_{ij} + \sum_l c_l} \prod_{i=1}^{n-1} [(-a_i)_{\sum_{i \neq j} b_{ij} + c_i}] (-a_n)_{\sum_{j \neq n} b_{nj}}}{(\sigma_n + D/2)_{2 \sum_{i>j} b_{ij} + \sum_l c_l}} \\
 &\times \prod_{i>j}^n \left[\frac{1}{b_{ij}!} \left(\frac{l_{ij}^2}{m_n^2} \right)^{b_{ij}} \right] \prod_{l=1}^{n-1} \left[\frac{1}{c_l!} \left(1 - \frac{m_l^2}{m_n^2} \right)^{c_l} \right]
 \end{aligned}
 \tag{32}$$

which was the result obtained by Davydychev in [23] using the MB approach. So, with the generating functional (18) as the starting point—the same which we have, in a previous paper [27] in the NDIM approach, used to show how to obtain a general formula to any scalar one-loop Feynman integral, in covariant gauges—we were able to reproduce a MB result, equation (32), and more, to present another formula (26), also valid for n -point scalar integrals.

3.1. n -point function via Feynman parametrization

Our final task in this paper is to show how to solve an n -point scalar integral using the FP technique. As far as we know there is no such result in the literature calculated using FP. Of

course, it has to be the same as that we obtained before using NDIM and Davydychev's [23] MB approaches.

We start with the function $F^{(n)}$

$$\begin{aligned} F^{(n)} &= F^{(n)}(a_i; l_i; m_i; x_0, x_n) \\ &= \frac{\Gamma(a_1 + \cdots + a_n)}{\Gamma(a_1) \cdots \Gamma(a_n)} \prod_{i=0}^{n-2} \left[\int_0^{x_i} dx_i (x_i - x_{i+1})^{a_{i+1}-1} \right] (x_{n-1} - x_n)^{a_n-1} \\ &\quad \times \int \frac{d^D k}{\left\{ \prod_{i=1}^n [(k - l_i)^2 - m_i^2] (x_{i-1} - x_i) \right\}^{\sum_{i=1}^n a_i}} \end{aligned} \quad (33)$$

where $a_i \geq 0$. This function for $x_0 = 1, x_n = 0$, represents the Feynman parametrization to the integral of type (20). The above integral in k can be evaluated using (15),

$$\begin{aligned} F^{(n)} &= \pi^{D/2} \frac{\Gamma(a_1 + \cdots + a_n - D/2) (x_0 - x_n)^{-D/2}}{\Gamma(a_1) \cdots \Gamma(a_n)} \\ &\quad \times \prod_{i=0}^{n-2} \left[\int_0^{x_i} dx_i (x_i - x_{i+1})^{a_{i+1}-1} \right] (x_{n-1} - x_n)^{a_n-1} \\ &\quad \times \left\{ \sum_{i=1}^n (l_i^2 - m_i^2) (x_{i-1} - x_i) - \frac{1}{x_0 - x_n} \left[\sum_{i=1}^n l_i (x_{i-1} - x_i) \right]^2 \right\}^{\sum_i a_i - D/2}. \end{aligned} \quad (34)$$

Using now

$$\sum_{i=1}^n l_i^2 (x_{i-1} - x_i) - \frac{\left[\sum_{i=1}^n l_i (x_{i-1} - x_i) \right]^2}{x_0 - x_n} = \sum_{i>j} \frac{(x_{i-1} - x_i)(x_{j-1} - x_j)}{x_0 - x_n} \left(\frac{l_{ij}^2}{m_n^2} \right) \quad (35)$$

and

$$\sum_{i=1}^{n-1} m_i^2 (x_i - x_{i-1}) = -m_n^2 (x_0 - x_n) + \sum_{i=1}^{n-1} (m_i^2 - m_n^2) (x_i - x_{i-1}) \quad (36)$$

and also performing the Taylor expansion of the argument of the above integral, we get

$$\begin{aligned} F^{(n)} &= \pi^{D/2} (-m_n^2)^{D/2 - \sum_i a_i} \frac{1}{\Gamma(a_1) \cdots \Gamma(a_n)} \sum_{b_{ij}=0}^{\infty} \sum_{c_l=0}^{\infty} (x_0 - x_n)^{-D/2 - 2 \sum_{i>j} b_{ij} - \sum_{l=1}^n c_l} \\ &\quad \times \Gamma \left(\sum a_i - D/2 + \sum_{i>j} b_{ij} + \sum_{l=1}^{n-1} c_l \right) \\ &\quad \times \prod_{i>j} \left[\frac{1}{b_{ij}!} \left(\frac{l_{ij}^2}{m_n^2} \right)^{b_{ij}} \right] \prod_{l=1}^{n-1} \left[\frac{1}{c_l!} \left(1 - \frac{m_l^2}{m_n^2} \right)^{c_l} \right] \\ &\quad \times \prod_{i=0}^{n-2} \left[\int_0^{x_i} dx_i (x_i - x_{i+1})^{g_{i+1}-1} \right] (x_{n-1} - x_n)^{g_n-1} \end{aligned} \quad (37)$$

where

$$g_i = a_i + \sum_{j \neq i} b_{ij} + c_i \quad i = 1, 2, \dots, n-1 \quad (38)$$

$$g_n = a_n + \sum_{j \neq n} b_{nj}. \tag{39}$$

The above integral can be evaluated with help of (16). If we take $x_0 = 1, x_n = 0, F^{(n)} = J^{(n)}$, we arrive at

$$\begin{aligned}
 J^{(n)} &= \pi^{D/2} (-m_n^2)^{D/2 - \sum_i a_i} \frac{\Gamma(\sum a_i - D/2)}{\Gamma(\sum a_i)} \\
 &\times \sum_{b_{ij}=0}^{\infty} \sum_{c_l=0}^{\infty} \frac{(\sum_{i>j} a_i - D/2)_{\sum_{i>j} b_{ij} + \sum_l c_l} \prod_{i=1}^{n-1} [(a_i)_{\sum_{i \neq j} b_{ij} + c_i}]}{(\sigma_n + D/2)_{2 \sum_{i>j} b_{ij} + \sum_l c_l}} \\
 &\times \prod_{i>j}^n \left[\frac{1}{b_{ij}!} \left(\frac{l_{ij}^2}{m_n^2} \right)^{b_{ij}} \right] \prod_{l=1}^{n-1} \left[\frac{1}{c_l!} \left(1 - \frac{m_l^2}{m_n^2} \right)^{c_l} \right]. \tag{40}
 \end{aligned}$$

This result is the same as that obtained in the previous subsection via NDIM in (32). This agreement shows that, at the one-loop level, the NDIM, Feynman parametrization and MB representation present the same results and are equivalent: all of them can be used to solve all scalar Feynman loop integrals at one-loop order, with general masses, arbitrary exponents of propagators and dimension.

4. Discussion and conclusion

So far we have made calculations in order to show that the same class of generating functionals can be used to reproduce MB results. Depending on which Taylor expansions one carries out, one can proceed along the NDIM or MB route. The final results will be, obviously, the same, given in terms of generalized hypergeometric functions, being the exponents of propagators and arbitrary spacetime dimension.

However, one could ask which of these two routes, if any, is the one where Feynman integrals become simpler to solve. The first point to observe is what are the tools one has to master in order to tackle such integrals in both approaches: contour integration, Cauchy theorem and Barnes lemmas for MB, and solving systems of algebraic equations for NDIM. So far, so good. Second, the results, despite being the same, have to be worked out one by one in the MB context. On the other hand, using NDIM and solving the system of algebraic equations gives all the possible solutions (generalized hypergeometric functions) for the Feynman integral in question. Grouping them is a straightforward task: linear independent functions have to be summed, each set is a possible result in a given region of convergence [26]. Third, the massless case needs to be known in the case of MB in order to tackle massive integrals, but not so in NDIM.

We can summarize both approaches in table 1. In step 3 one can proceed as we have done in this paper, expanding the exponential, or as we did in our previous works taking a Taylor expansion for each argument of the exponential. The final step, 11, is to be understood in the following manner: in order to obtain all possible generalized hypergeometric functions (which come in NDIM) using MB, one has to repeat the above procedure choosing another sequence of contours, we mean for instance left-left-right-left-right and another one left-left-right-right-right, these two can give, in principle, distinct generalized hypergeometric series. Some of them will of course result in zero, since there can be no poles inside the contours. These are also contained in the NDIM approach, since some determinants can vanish, a much simpler calculation that can be implemented in software such as Mathematica.

Table 1. Schematically the steps one should follow in order to calculate loop integrals using MB and NDIM approaches.

Step	MB	NDIM
1	Generating functional	Generating functional
2	Solve it	Solve it
3	Taylor expand (whole)	Taylor expand (each or whole)
4	Mellin transform	Project powers
5	Compare term by term	Compare term by term
6	Solve it for the integral	Solve it for the integral
7	Result: parametric integrals	Result: system of algebraic equations
8	Choose the contour: left or right	Elementary techniques
9	Cauchy theorem	Use the results of the systems
10	and Barnes' lemmas	Analytically continue to positive D
11	One has one final result among several	One has all the series (final results)

The textbook technique, FP, can be made simpler if one introduces two extra parameters x_0 and x_n , and takes series expansions in the parameters $(x_0 - x_1), (x_1 - x_2), \dots, (x_{n-1} - x_n)$. At the end of the day, one makes $x_0 = 0, x_n = 1$ and uses the well-known beta function integral representation. Then, the remaining expression is the result written as a generalized hypergeometric function.

4.1. Conclusion

We have shown that the negative-dimensional integration method (NDIM), Feynman parametrization (FP) and the Mellin–Barnes (MB) approach to scalar Feynman loop integrals, at the one-loop level, give the same results. It depends only on how one chooses to Taylor expand the generating functional (18). We present detailed calculations for two-point scalar integrals, with arbitrary masses, exponents of propagators and spacetime dimension (in covariant gauges). Then we tackle a general scalar n -point integral, with different masses, and show that the general formulae of Davydychev [23] and ours [27] agree, as well as another one obtained via FP worked out, as far as we know, for the first time. It is our opinion, however, that NDIM is simpler than MB, since all the possible results for the integral in question are obtained simultaneously, and in MB they must be calculated one by one, or through analytic continuation formulae, if such formulae were known, depending on the hypergeometric functions. FP is also a very powerful technique if one introduces two extra parameters and takes Taylor expansions properly. In doing so, FP can become even simpler than NDIM, since one obtains the full result and does not have the drawback of searching among a large number of possible solutions.

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